Nearly Optimal Embeddings of Flat Tori

Technical Proofs

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Joint work with Ishan Agarwal and Oded Regev

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Notations & Definitions

- lattice $\mathcal{L} = \mathbf{B}\mathbb{Z}^n$
  - minimum distance $\lambda_1(\mathcal{L}) := \min\{r > 0 : \text{rank}(\mathcal{L} \cap \mathcal{B}_r) \geq 1\}$
  - *covering radius* $\mu(\mathcal{L}) := \max_{\mathbf{x} \in \text{span}(\mathcal{L})} \text{dist}(\mathbf{x}, \mathcal{L})$
  - *quotient lattice* $\mathcal{L}/\mathcal{L}' := \pi^{\perp}_{\text{span}(\mathcal{L}')}(\mathcal{L})$, for $\mathcal{L}' \subset \mathcal{L}$
- (flat) torus $\mathbb{R}^n/\mathcal{L}$
  - torus metric: $\text{dist}_{\mathbb{R}^n/\mathcal{L}}(\mathbf{x} + \mathcal{L}, \mathbf{y} + \mathcal{L}) = \text{dist}(\mathbf{x} - \mathbf{y}, \mathcal{L})$ (write $\text{dist}_{\mathbb{R}^n/\mathcal{L}}(\mathbf{x}, \mathbf{y})$ for simplicity)
  - distortion of (injective) embedding f: expansion/contraction; expansion: $\sup_{\mathbf{x},\mathbf{y}} \frac{\text{dist}(f(\mathbf{x}), f(\mathbf{y}))}{\text{dist}_{\mathbb{R}^n/\mathcal{L}}(\mathbf{x},\mathbf{y})}$, contraction: $\inf \ldots$
- Goal: embed $\mathbb{R}^n/\mathcal{L}$ into L_2, with distortion $O(\sqrt{n \log n})$

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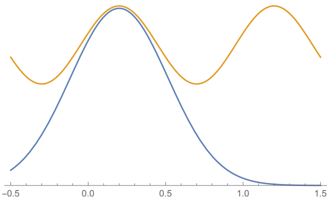
The HR10 Embedding

The HR10 Embedding $H_{\mathcal{L},k}(\mathbf{x})$ maps $\mathbf{x} \in \mathbb{R}^n/\mathcal{L}$ to a k-tuple (in ℓ_2) of Gaussians centered at $\mathbf{x}$ with certain variances and coefficients (determined by the "scale" $\lambda_1(\mathcal{L})$).

Wrapping the Gaussians:

- strictly speaking inputs to $H_{\mathcal{L},k}$ should be $\mathbf{x} + \mathcal{L} \in \mathbb{R}^n/\mathcal{L}$
- consequently for $H_{\mathcal{L},k}$ to be well defined, the output Gaussians should be "wrapped around," i.e., be the sum of all copies centered at $\mathbf{x} + \mathcal{L}$, and live in $L_2(\mathbb{R}^n/\mathcal{L})$ instead of $L_2(\mathbb{R}^n)$



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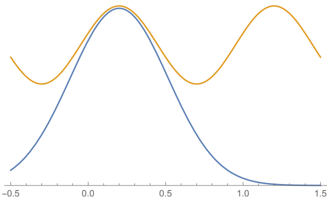

Distortion of The HR10 Embedding

$H_{\mathcal{L},k}$ has distortion $O(\sqrt{nk})$:

- expansion: $\leq \sqrt{\pi k}$
- contraction: $\geq \sqrt{c_H/n}$, where c_H is absolute constant
- caveat for contraction: saturation at $2^{k-1}\lambda_1(\mathcal{L})$, i.e., only have contraction w.r.t. $\min\left(\mathrm{dist}_{\mathbb{R}^n/\mathcal{L}}(\mathbf{x},\mathbf{y}), 2^{k-1}\lambda_1(\mathcal{L})\right)$

Choices of k in HR10:

- $k = O(\log \frac{\mu(\mathcal{L})}{\lambda_1(\mathcal{L})})$: distortion $O\left(\sqrt{n \log \frac{\mu(\mathcal{L})}{\lambda_1(\mathcal{L})}}\right)$
- $k = O(\log n)$: distortion $O(\sqrt{n \log n})$, while requiring $\mu(\mathcal{L}) \leq \mathrm{poly}(n) \cdot \lambda_1(\mathcal{L})$

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The Partitioning Embedding

- given *good filtration* $\mathcal{F} : \{\vec{0}\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m = \mathcal{L}$
- define projections $\pi_{\mathcal{F}}^{=j} := \pi_{\mathsf{span}(\mathcal{L}_j/\mathcal{L}_{j-1})}$ for $j \in [m]$ (and analogously $\pi_{\mathcal{F}}^{\geq j}, \pi_{\mathcal{F}}^{>j}, \pi_{\mathcal{F}}^{<j}, \pi_{\mathcal{F}}^{\leq j}$); this gives an orthogonal decomposition of the entire space
- define the *compressed projections* $E_{\mathcal{F},\alpha}^{(j)} := \sum_{i=j}^{m} \alpha^{i-j} \pi_{\mathcal{F}}^{=i}$ for $j \in [m]$, and the overall partitioning embedding $E_{\mathcal{F},\alpha}$ to be the tuple $(E_{\mathcal{F},\alpha}^{(1)}, \ldots, E_{\mathcal{F},\alpha}^{(m)})$ (in ℓ_2)
- note that $E_{\mathcal{F},\alpha}^{(j)}(\mathcal{L})$ is not dense as long as $\alpha > 0$, and thus $E_{\mathcal{F},\alpha}^{(j)}(\mathbb{R}^n/\mathcal{L})$ gives a valid torus

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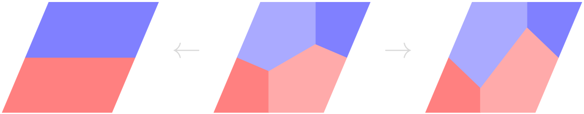
Where does Distortion Come from?

- want to embed $\text{dist}_{\mathbb{R}^n/\mathcal{L}}(\mathbf{x}, \mathbf{y}) = \text{dist}(\mathbf{x} - \mathbf{y}, \mathcal{L})$
  (for simplicity suppose $\mathbf{y} = \vec{0}$)
- let $\mathbf{v} \in \mathcal{L}$ be a closest lattice vector (CV) to $\mathbf{x}$;
  then $\text{dist}(\mathbf{x}, \mathcal{L}) = \|\mathbf{x} - \mathbf{v}\|$
- want $\text{dist}(E^{(j)}_{\mathcal{F},\alpha}(\mathbf{x}), E^{(j)}_{\mathcal{F},\alpha}(\mathcal{L})) = \|E^{(j)}_{\mathcal{F},\alpha}(\mathbf{x} - \mathbf{v})\|$ so that they add up to $\Theta(1) \cdot \|\mathbf{x} - \mathbf{v}\|$ and there is constant distortion
- however $E^{(j)}_{\mathcal{F},\alpha}(\mathbf{v})$ is not necessarily CV to $E^{(j)}_{\mathcal{F},\alpha}(\mathbf{x})$ due to:
  1. projection (left figure: project onto y-direction)
  2. compression (right figure: compress y-direction by $\alpha = 1/2$)

  both distorting the geometry



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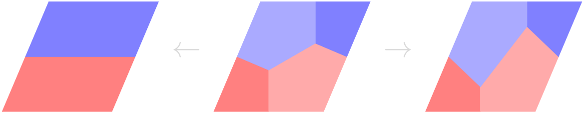

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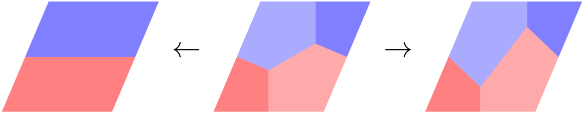

Expansion of The Partitioning Embedding

Although CV could change in each compressed projection, this only leads to shorter embedded distance and does not harm expansion.

The expansion is easily $\leq \sqrt{\frac{1}{1-\alpha^2}}$ thanks to the geometric series (and square root due to using ℓ_2 tuple).

Contraction of The Partitioning Embedding: Act 1

- want to prove constant contraction
- let j_1 be the last index where CV changes
- we know the part $\|\pi_{\mathcal{F}}^{>j_1}(\mathbf{x}-\mathbf{v})\|$ is "captured" by $E_{\mathcal{F},\alpha}^{(>j_1)}$
- if this part is already a constant fraction of $\|\mathbf{x}-\mathbf{v}\|$ then we get constant contraction
- so from now on suppose, say, $\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\|^2 > \frac{1}{2}\|\mathbf{x}-\mathbf{v}\|^2$
- we also know $\|E_{\mathcal{F},\alpha}^{(j_1)}(\mathbf{x}-\mathbf{v})\| \geq \frac{1}{2}\lambda_1(E_{\mathcal{F},\alpha}^{(j_1)}(\mathcal{L}))$, due to change of CV (by triangle ineq., $\|E_{\mathcal{F},\alpha}^{(j_1)}(\mathbf{v}-\mathbf{v}^{(j_1)})\| \leq 2\|E_{\mathcal{F},\alpha}^{(j_1)}(\mathbf{x}-\mathbf{v})\|$, where $E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{v}^{(j)})$ is CV to $E_{\mathcal{F},\alpha}^{(j)}(\mathbf{x})$)

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Contraction of The Partitioning Embedding: Act 2

- suffice to find $j_0 \leq j_1$ s.t. $\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v}')\|$ captures $\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\|$, where $\mathbf{v}' = \mathbf{v}^{(j_0)}$ ($E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{v}')$ is CV to $E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x})$) (w.l.o.g. $\|\pi_{\mathcal{F}}^{< j_0}(\mathbf{x}-\mathbf{v}')\| \leq \mu(\mathcal{L}_{j_0-1})$)
- try to bound $\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v}')\|^2 = \sum_{i=j_0}^{m} \alpha^{2(i-j_0)} \|\pi_{\mathcal{F}}^{=i}(\mathbf{x}-\mathbf{v}')\|^2$
- truncate the sum at some $j_2 \geq j_1$ to handle the exponential factor: $\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v}')\|^2 \geq \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi_{\mathcal{F}}^{=i}(\mathbf{x}-\mathbf{v}')\|^2$
- note that $\sum_{i=j_0}^{j_2} \|\pi_{\mathcal{F}}^{=i}(\cdot)\|^2 = \|\cdot\|^2 - \|\pi_{\mathcal{F}}^{< j_0}(\cdot)\|^2 - \|\pi_{\mathcal{F}}^{> j_2}(\cdot)\|^2$
  1. $\|\mathbf{x}-\mathbf{v}'\|^2 \geq \|\mathbf{x}-\mathbf{v}\|^2$ as $\mathbf{v}$ is CV
  2. $\|\pi_{\mathcal{F}}^{< j_0}(\mathbf{x}-\mathbf{v}')\| \leq \mu(\mathcal{L}_{j_0-1})$ for free;
     want $\mu(\mathcal{L}_{j_0-1}) \leq \frac{1}{4}\lambda_1(E_{\mathcal{F},\alpha}^{(j_1)}(\mathcal{L}))$;
     then $\|\pi_{\mathcal{F}}^{< j_0}(\mathbf{x}-\mathbf{v}')\| \leq \frac{1}{2}\|\mathbf{x}-\mathbf{v}\|$
  3. hopefully $\|\pi_{\mathcal{F}}^{> j_2}(\mathbf{x}-\mathbf{v}')\| = \|\pi_{\mathcal{F}}^{> j_2}(\mathbf{x}-\mathbf{v})\|$ ($< \frac{1}{\sqrt{2}}\|\mathbf{x}-\mathbf{v}\|$)

Contraction of The Partitioning Embedding: Act 2

- suffice to find $j_0 \le j_1$ s.t. $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|$ captures $\|\pi^{\le j_1}_{\mathcal{F}}(\mathbf{x}-\mathbf{v})\|$, where $\mathbf{v}' = \mathbf{v}^{(j_0)}$ ($E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{v}')$ is CV to $E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x})$) (w.l.o.g. $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \mu(\mathcal{L}_{j_0-1})$)
- try to bound $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|^2 = \sum_{i=j_0}^{m} \alpha^{2(i-j_0)} \|\pi^{=i}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\|^2$
- truncate the sum at some $j_2 \ge j_1$ to handle the exponential factor: $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|^2 \ge \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi^{=i}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\|^2$
- note that $\sum_{i=j_0}^{j_2} \|\pi^{=i}_{\mathcal{F}}(\cdot)\|^2 = \|\cdot\|^2 - \|\pi^{<j_0}_{\mathcal{F}}(\cdot)\|^2 - \|\pi^{>j_2}_{\mathcal{F}}(\cdot)\|^2$
  1. $\|\mathbf{x}-\mathbf{v}'\|^2 \ge \|\mathbf{x}-\mathbf{v}\|^2$ as $\mathbf{v}$ is CV
  2. $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \mu(\mathcal{L}_{j_0-1})$ for free;
     want $\mu(\mathcal{L}_{j_0-1}) \le \frac{1}{4}\lambda_1(E^{(j_1)}_{\mathcal{F},\alpha}(\mathcal{L}))$;
     then $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \frac{1}{2}\|\mathbf{x}-\mathbf{v}\|$
  3. hopefully $\|\pi^{>j_2}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| = \|\pi^{>j_2}_{\mathcal{F}}(\mathbf{x}-\mathbf{v})\|$ ($< \frac{1}{\sqrt{2}}\|\mathbf{x}-\mathbf{v}\|$)

Contraction of The Partitioning Embedding: Act 2

- suffice to find $j_0 \le j_1$ s.t. $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|$ captures $\|\pi^{\le j_1}_{\mathcal{F}}(\mathbf{x}-\mathbf{v})\|$, where $\mathbf{v}' = \mathbf{v}^{(j_0)}$ ($E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{v}')$ is CV to $E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x})$) (w.l.o.g. $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \mu(\mathcal{L}_{j_0-1})$)
- try to bound $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|^2 = \sum_{i=j_0}^{m} \alpha^{2(i-j_0)} \|\pi^{=i}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\|^2$
- truncate the sum at some $j_2 \ge j_1$ to handle the exponential factor: $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|^2 \ge \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi^{=i}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\|^2$
- note that $\sum_{i=j_0}^{j_2} \|\pi^{=i}_{\mathcal{F}}(\cdot)\|^2 = \|\cdot\|^2 - \|\pi^{<j_0}_{\mathcal{F}}(\cdot)\|^2 - \|\pi^{>j_2}_{\mathcal{F}}(\cdot)\|^2$
  1. $\|\mathbf{x}-\mathbf{v}'\|^2 \ge \|\mathbf{x}-\mathbf{v}\|^2$ as $\mathbf{v}$ is CV
  2. $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \mu(\mathcal{L}_{j_0-1})$ for free;
     want $\mu(\mathcal{L}_{j_0-1}) \le \frac{1}{4}\lambda_1(E^{(j_1)}_{\mathcal{F},\alpha}(\mathcal{L}))$;
     then $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \frac{1}{2}\|\mathbf{x}-\mathbf{v}\|$
  3. hopefully $\|\pi^{>j_2}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| = \|\pi^{>j_2}_{\mathcal{F}}(\mathbf{x}-\mathbf{v})\|$ ($< \frac{1}{\sqrt{2}}\|\mathbf{x}-\mathbf{v}\|$)

Contraction of The Partitioning Embedding: Act 2

- suffice to find $j_0 \le j_1$ s.t. $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|$ captures $\|\pi^{\le j_1}_{\mathcal{F}}(\mathbf{x}-\mathbf{v})\|$, where $\mathbf{v}' = \mathbf{v}^{(j_0)}$ ($E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{v}')$ is CV to $E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x})$) (w.l.o.g. $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \mu(\mathcal{L}_{j_0-1})$)
- try to bound $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|^2 = \sum_{i=j_0}^{m} \alpha^{2(i-j_0)} \|\pi^{=i}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\|^2$
- truncate the sum at some $j_2 \ge j_1$ to handle the exponential factor: $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|^2 \ge \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi^{=i}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\|^2$
- note that $\sum_{i=j_0}^{j_2} \|\pi^{=i}_{\mathcal{F}}(\cdot)\|^2 = \|\cdot\|^2 - \|\pi^{<j_0}_{\mathcal{F}}(\cdot)\|^2 - \|\pi^{>j_2}_{\mathcal{F}}(\cdot)\|^2$
    1. $\|\mathbf{x}-\mathbf{v}'\|^2 \ge \|\mathbf{x}-\mathbf{v}\|^2$ as $\mathbf{v}$ is CV
    2. $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \mu(\mathcal{L}_{j_0-1})$ for free;
       want $\mu(\mathcal{L}_{j_0-1}) \le \frac{1}{4}\lambda_1(E^{(j_1)}_{\mathcal{F},\alpha}(\mathcal{L}))$;
       then $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \frac{1}{2}\|\mathbf{x}-\mathbf{v}\|$
    3. hopefully $\|\pi^{>j_2}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| = \|\pi^{>j_2}_{\mathcal{F}}(\mathbf{x}-\mathbf{v})\|$ ($< \frac{1}{\sqrt{2}}\|\mathbf{x}-\mathbf{v}\|$)

Contraction of The Partitioning Embedding: Act 2

- suffice to find $j_0 \leq j_1$ s.t. $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|$ captures $\|\pi^{\leq j_1}_{\mathcal{F}}(\mathbf{x}-\mathbf{v})\|$, where $\mathbf{v}' = \mathbf{v}^{(j_0)}$ ($E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{v}')$ is CV to $E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x})$) (w.l.o.g. $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \leq \mu(\mathcal{L}_{j_0-1})$)
- try to bound $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|^2 = \sum_{i=j_0}^{m} \alpha^{2(i-j_0)} \|\pi^{=i}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\|^2$
- truncate the sum at some $j_2 \geq j_1$ to handle the exponential factor: $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|^2 \geq \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi^{=i}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\|^2$
- note that $\sum_{i=j_0}^{j_2} \|\pi^{=i}_{\mathcal{F}}(\cdot)\|^2 = \|\cdot\|^2 - \|\pi^{<j_0}_{\mathcal{F}}(\cdot)\|^2 - \|\pi^{>j_2}_{\mathcal{F}}(\cdot)\|^2$
  1. $\|\mathbf{x}-\mathbf{v}'\|^2 \geq \|\mathbf{x}-\mathbf{v}\|^2$ as $\mathbf{v}$ is CV
  2. $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \leq \mu(\mathcal{L}_{j_0-1})$ for free;
     want $\mu(\mathcal{L}_{j_0-1}) \leq \frac{1}{4}\lambda_1(E^{(j_1)}_{\mathcal{F},\alpha}(\mathcal{L}))$;
     then $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \leq \frac{1}{2}\|\mathbf{x}-\mathbf{v}\|$
  3. hopefully $\|\pi^{>j_2}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| = \|\pi^{>j_2}_{\mathcal{F}}(\mathbf{x}-\mathbf{v})\|$ ($< \frac{1}{\sqrt{2}}\|\mathbf{x}-\mathbf{v}\|$)

Contraction of The Partitioning Embedding: Act 2

- suffice to find $j_0 \le j_1$ s.t. $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|$ captures $\|\pi^{\le j_1}_{\mathcal{F}}(\mathbf{x}-\mathbf{v})\|$, where $\mathbf{v}' = \mathbf{v}^{(j_0)}$ ($E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{v}')$ is CV to $E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x})$) (w.l.o.g. $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \mu(\mathcal{L}_{j_0-1})$)
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- truncate the sum at some $j_2 \ge j_1$ to handle the exponential factor: $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x}-\mathbf{v}')\|^2 \ge \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi^{=i}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\|^2$
- note that $\sum_{i=j_0}^{j_2} \|\pi^{=i}_{\mathcal{F}}(\cdot)\|^2 = \|\cdot\|^2 - \|\pi^{<j_0}_{\mathcal{F}}(\cdot)\|^2 - \|\pi^{>j_2}_{\mathcal{F}}(\cdot)\|^2$
  1. $\|\mathbf{x}-\mathbf{v}'\|^2 \ge \|\mathbf{x}-\mathbf{v}\|^2$ as $\mathbf{v}$ is CV
  2. $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \mu(\mathcal{L}_{j_0-1})$ for free;
     want $\mu(\mathcal{L}_{j_0-1}) \le \frac{1}{4}\lambda_1(E^{(j_1)}_{\mathcal{F},\alpha}(\mathcal{L}))$;
     then $\|\pi^{<j_0}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| \le \frac{1}{2}\|\mathbf{x}-\mathbf{v}\|$
  3. hopefully $\|\pi^{>j_2}_{\mathcal{F}}(\mathbf{x}-\mathbf{v}')\| = \|\pi^{>j_2}_{\mathcal{F}}(\mathbf{x}-\mathbf{v})\|$ $(< \frac{1}{\sqrt{2}}\|\mathbf{x}-\mathbf{v}\|)$

Contraction of The Partitioning Embedding: Act 2

- suffice to find $j_0 \le j_1$ s.t. $\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v}')\|$ captures $\|\pi_{\mathcal{F}}^{\le j_1}(\mathbf{x}-\mathbf{v})\|$, where $\mathbf{v}' = \mathbf{v}^{(j_0)}$ ($E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{v}')$ is CV to $E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x})$) (w.l.o.g. $\|\pi_{\mathcal{F}}^{<j_0}(\mathbf{x}-\mathbf{v}')\| \le \mu(\mathcal{L}_{j_0-1})$)
- try to bound $\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v}')\|^2 = \sum_{i=j_0}^{m} \alpha^{2(i-j_0)} \|\pi_{\mathcal{F}}^{=i}(\mathbf{x}-\mathbf{v}')\|^2$
- truncate the sum at some $j_2 \ge j_1$ to handle the exponential factor: $\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v}')\|^2 \ge \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi_{\mathcal{F}}^{=i}(\mathbf{x}-\mathbf{v}')\|^2$
- note that $\sum_{i=j_0}^{j_2} \|\pi_{\mathcal{F}}^{=i}(\cdot)\|^2 = \|\cdot\|^2 - \|\pi_{\mathcal{F}}^{<j_0}(\cdot)\|^2 - \|\pi_{\mathcal{F}}^{>j_2}(\cdot)\|^2$
    1. $\|\mathbf{x}-\mathbf{v}'\|^2 \ge \|\mathbf{x}-\mathbf{v}\|^2$ as $\mathbf{v}$ is CV
    2. $\|\pi_{\mathcal{F}}^{<j_0}(\mathbf{x}-\mathbf{v}')\| \le \mu(\mathcal{L}_{j_0-1})$ for free;
       want $\mu(\mathcal{L}_{j_0-1}) \le \frac{1}{4}\lambda_1(E_{\mathcal{F},\alpha}^{(j_1)}(\mathcal{L}))$;
       then $\|\pi_{\mathcal{F}}^{<j_0}(\mathbf{x}-\mathbf{v}')\| \le \frac{1}{2}\|\mathbf{x}-\mathbf{v}\|$
    3. hopefully $\|\pi_{\mathcal{F}}^{>j_2}(\mathbf{x}-\mathbf{v}')\| = \|\pi_{\mathcal{F}}^{>j_2}(\mathbf{x}-\mathbf{v})\|$ ($< \frac{1}{\sqrt{2}}\|\mathbf{x}-\mathbf{v}\|$)

Contraction of The Partitioning Embedding: Act 2.5

- "hopefully $\|\pi_{\mathcal{F}}^{>j_2}(\mathbf{x}-\mathbf{v}')\| = \|\pi_{\mathcal{F}}^{>j_2}(\mathbf{x}-\mathbf{v})\|$"
  - suffice to show $\pi_{\mathcal{F}}^{>j_2}(\mathbf{v}') = \pi_{\mathcal{F}}^{>j_2}(\mathbf{v})$, or $E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathbf{v}') = E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathbf{v})$
  - if not, they are distant: $\|E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathbf{v}-\mathbf{v}')\| \geq \lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))$
  - note that by algebra, $\|E_{\mathcal{F},\alpha}^{(j_0)}(\cdot)\| \geq \alpha^{j_2+1-j_0}\|E_{\mathcal{F},\alpha}^{(j_2+1)}(\cdot)\|$
  - hence

$$\begin{aligned}\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\| > \frac{1}{\sqrt{2}}\|\mathbf{x}-\mathbf{v}\| &\geq \frac{1}{\sqrt{2}}\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v})\| \\ &\geq \frac{1}{2\sqrt{2}}\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{v}-\mathbf{v}')\| \\ &\geq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}}\lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))\end{aligned}$$

  - on the other hand $\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\| \leq \mu(\mathcal{L}_{j_1})$;
    so want $\mu(\mathcal{L}_{j_1}) \leq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}}\lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))$ for contradiction

Contraction of The Partitioning Embedding: Act 2.5

- "hopefully $\|\pi_{\mathcal{F}}^{>j_2}(\mathbf{x}-\mathbf{v}')\| = \|\pi_{\mathcal{F}}^{>j_2}(\mathbf{x}-\mathbf{v})\|$"
  - suffice to show $\pi_{\mathcal{F}}^{>j_2}(\mathbf{v}') = \pi_{\mathcal{F}}^{>j_2}(\mathbf{v})$, or $E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathbf{v}') = E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathbf{v})$
  - if not, they are distant: $\|E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathbf{v}-\mathbf{v}')\| \geq \lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))$
  - note that by algebra, $\|E_{\mathcal{F},\alpha}^{(j_0)}(\cdot)\| \geq \alpha^{j_2+1-j_0}\|E_{\mathcal{F},\alpha}^{(j_2+1)}(\cdot)\|$
  - hence

$$\begin{aligned}\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\| > \frac{1}{\sqrt{2}}\|\mathbf{x}-\mathbf{v}\| &\geq \frac{1}{\sqrt{2}}\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v})\| \\ &\geq \frac{1}{2\sqrt{2}}\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{v}-\mathbf{v}')\| \\ &\geq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}}\lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))\end{aligned}$$

  - on the other hand $\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\| \leq \mu(\mathcal{L}_{j_1})$;
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Contraction of The Partitioning Embedding: Act 2.5

- "hopefully $\|\pi_{\mathcal{F}}^{>j_2}(\mathbf{x}-\mathbf{v}')\| = \|\pi_{\mathcal{F}}^{>j_2}(\mathbf{x}-\mathbf{v})\|$"
  - suffice to show $\pi_{\mathcal{F}}^{>j_2}(\mathbf{v}') = \pi_{\mathcal{F}}^{>j_2}(\mathbf{v})$, or $E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathbf{v}') = E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathbf{v})$
  - if not, they are distant: $\|E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathbf{v}-\mathbf{v}')\| \geq \lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))$
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$$
\begin{aligned}
\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\| > \frac{1}{\sqrt{2}}\|\mathbf{x}-\mathbf{v}\| &\geq \frac{1}{\sqrt{2}}\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v})\| \\
&\geq \frac{1}{2\sqrt{2}}\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{v}-\mathbf{v}')\| \\
&\geq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}}\lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))
\end{aligned}
$$

  - on the other hand $\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\| \leq \mu(\mathcal{L}_{j_1})$;
    so want $\mu(\mathcal{L}_{j_1}) \leq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}}\lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))$ for contradiction

Contraction of The Partitioning Embedding: Act 2.5

- "hopefully $\|\pi_{\mathcal{F}}^{>j_2}(\mathbf{x}-\mathbf{v}')\| = \|\pi_{\mathcal{F}}^{>j_2}(\mathbf{x}-\mathbf{v})\|$"
  - suffice to show $\pi_{\mathcal{F}}^{>j_2}(\mathbf{v}') = \pi_{\mathcal{F}}^{>j_2}(\mathbf{v})$, or $E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathbf{v}') = E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathbf{v})$
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  - hence

$$\begin{aligned}
\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\| > \frac{1}{\sqrt{2}}\|\mathbf{x}-\mathbf{v}\| &\geq \frac{1}{\sqrt{2}}\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v})\| \\
&\geq \frac{1}{2\sqrt{2}}\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{v}-\mathbf{v}')\| \\
&\geq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}}\lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))
\end{aligned}$$

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Contraction of The Partitioning Embedding: Act 3

- already manage to capture $\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\|$, even the entire $\|\mathbf{x}-\mathbf{v}\|$, by $\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v}')\|$?
- need to consider saturation of HR10, i.e., can only use $\min(\|E_{\mathcal{F},\alpha}^{(j)}(\mathbf{x}-\mathbf{v}^{(j)})\|, \mathrm{poly}(n)\cdot\lambda_1(E_{\mathcal{F},\alpha}^{(j)}(\mathcal{L})))$ for each j
- for $E_{\mathcal{F},\alpha}^{(>j_1)}$, they still capture $\|\pi_{\mathcal{F}}^{>j_1}(\mathbf{x}-\mathbf{v})\|$ as long as $\mu(\mathcal{L}_j)\leq \mathrm{poly}(n)\cdot\lambda_1(E_{\mathcal{F},\alpha}^{(j)}(\mathcal{L}))$
- for $E_{\mathcal{F},\alpha}^{(j_0)}$ (to capture $\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\|$), want $\mu(\mathcal{L}_{j_1})\leq \mathrm{poly}(n)\cdot\lambda_1(E_{\mathcal{F},\alpha}^{(j_0)}(\mathcal{L}))$

Finally we have $\Theta(1)$ contraction (considering saturation of HR10), and thus $\Theta(1)$ distortion of the partitioning embedding, and thus $O(\sqrt{n\log n})$ overall distortion after composing with the HR10 embedding.

Contraction of The Partitioning Embedding: Act 3

- already manage to capture $\|\pi_{\mathcal{F}}^{\le j_1}(\mathbf{x}-\mathbf{v})\|$, even the entire $\|\mathbf{x}-\mathbf{v}\|$, by $\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v}')\|$?
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- already manage to capture $\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\|$, even the entire $\|\mathbf{x}-\mathbf{v}\|$, by $\|E_{\mathcal{F},\alpha}^{(j_0)}(\mathbf{x}-\mathbf{v}')\|$?
- need to consider saturation of HR10, i.e., can only use $\min\left(\|E_{\mathcal{F},\alpha}^{(j)}(\mathbf{x}-\mathbf{v}^{(j)})\|, \mathrm{poly}(n)\cdot\lambda_1(E_{\mathcal{F},\alpha}^{(j)}(\mathcal{L}))\right)$ for each j
- for $E_{\mathcal{F},\alpha}^{(>j_1)}$, they still capture $\|\pi_{\mathcal{F}}^{>j_1}(\mathbf{x}-\mathbf{v})\|$ as long as $\mu(\mathcal{L}_j) \leq \mathrm{poly}(n)\cdot\lambda_1(E_{\mathcal{F},\alpha}^{(j)}(\mathcal{L}))$
- for $E_{\mathcal{F},\alpha}^{(j_0)}$ (to capture $\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x}-\mathbf{v})\|$), want $\mu(\mathcal{L}_{j_1}) \leq \mathrm{poly}(n)\cdot\lambda_1(E_{\mathcal{F},\alpha}^{(j_0)}(\mathcal{L}))$

Finally we have $\Theta(1)$ contraction (considering saturation of HR10), and thus $\Theta(1)$ distortion of the partitioning embedding, and thus $O(\sqrt{n\log n})$ overall distortion after composing with the HR10 embedding.

Good Filtration

During the contraction proof we assumed:
$\mu(\mathcal{L}_j) \leq \text{poly}(n) \cdot \lambda_1(E^{(j)}_{\mathcal{F},\alpha}(\mathcal{L}))$, $\mu(\mathcal{L}_{j_0-1}) \leq \frac{1}{4}\,\lambda_1(E^{(j_1)}_{\mathcal{F},\alpha}(\mathcal{L}))$,
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These reduce to (β, γ)-*filtration*:

- $\mu(\mathcal{L}_j/\mathcal{L}_{j-1}) \leq \beta \cdot \lambda_1(\mathcal{L}_j/\mathcal{L}_{j-1})$
- $\lambda_1(\mathcal{L}_{j+1}/\mathcal{L}_j) \geq \gamma \cdot \lambda_1(\mathcal{L}_j/\mathcal{L}_{j-1})$
- i.e., separated scales!

(along with mild enough compression $\alpha \geq 1/\gamma$)

$(\gamma\sqrt{n}, \gamma)$-filtration can be achieved using *Korkine–Zolotarev basis*.
The idea is intuitive: to group shortest bases into one sublattice until reaching a next scale that is γ times larger.

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Open Questions, Extended

- Finite dimensional embeddings
  - via discretization?
  - trade-off between distortion and dimensionality
- Efficiently computable embeddings
  - Korkine–Zolotarev basis is hard to compute
  - necessary for algorithmic applications, if any :|
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  - e.g. $\Theta(1)$ distortion for $\mathbb{Z}^n$
  - potentially involving $\lambda_1(\mathcal{L})$ and $\mu(\mathcal{L})$
  - lower bound: $\Omega(\lambda_1(\mathcal{L}^*) \cdot \mu(\mathcal{L})/\sqrt{n}) \geq \Omega(\frac{\lambda_1(\mathcal{L}^*)}{\mu(\mathcal{L}^*)} \cdot \sqrt{n})$
  - by having "adaptive" compression factor α?
- Embedding into L_p instead of L_2
  - for L_1, same lower bound
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